SL Paper 2

a. (i) Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)) = 2\sec^3\theta$. [5]

(ii) Hence write down $\int \sec^3\theta \mathrm{d}\theta$.

b. Consider the differential equation $(1 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 1 + x^2$ given that $y = 1$ when $x = 0$. [24]

(i) Use Euler's method with a step length of 0.1 to find an approximate value for y when $x = 0.3$.

(ii) Find an integrating factor for determining the exact solution of the differential equation.

(iii) Find the solution of the equation in the form $y = f(x)$.

(iv) To how many significant figures does the approximation found in part (i) agree with the exact value of y when $x = 0.3$?

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b. (i) Show that the improper integral $\int_0^\infty \frac{1}{x^2+1}\mathrm{d}x$ is convergent. [6]

(ii) Use the integral test to deduce that the series $\sum\limits_{n=0}^{\infty} \frac{1}{n^2+1}$ is convergent, giving reasons why this test can be applied.

c. (i) Show that the series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ is convergent. [6]

(ii) If the sum of the above series is S, show that $\frac{3}{5} < S < \frac{2}{3}$.

d. For the series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n^2+1}$ [6]

(i) determine the radius of convergence;

(ii) determine the interval of convergence using your answers to (b) and (c).

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = 2\sec^2 x,\ 0 \leqslant x < \frac{\pi}{2}, \text{ given that } y = 1 \text{ when } x = 0.$$

a.i. By considering integration as the reverse of differentiation, show that for [4]

$0 \leqslant x < \frac{\pi}{2}$

$$\int \sec x \mathrm{d}x = \ln(\sec x + \tan x) + C.$$

a.ii. Hence, using integration by parts, show that [4]

$$\int \sec^3 x \mathrm{d}x = \frac{1}{2}(\sec x\tan x + \ln(\sec x + \tan x)) + C.$$

b.i. Find an integrating factor and hence solve the differential equation, giving your answer in the form $y = f(x)$. [9]

b.ii. Starting with the differential equation, show that [3]

$$\frac{d^2y}{dx^2} + y = 2\sec^2 x \tan x.$$

b.iii. Hence, by using your calculator to draw two appropriate graphs or otherwise, find the x-coordinate of the point of inflection on the graph of $y = f(x)$. [4]

Consider the differential equation $\frac{dy}{dx} = 2x + y - 1$ with boundary condition $y = 1$ when $x = 0$.

a. Using Euler's method with increments of 0.2, find an approximate value for y when $x = 1$. [5]

b. Explain how Euler's method could be improved to provide a better approximation. [1]

c. Solve the differential equation to find an exact value for y when $x = 1$. [9]

d. (i) Find the first three non-zero terms of the Maclaurin series for y. [5]

(ii) Hence find an approximate value for y when $x = 1$.

The function $f(x)$ is defined by the series $f(x) = 1 + \frac{(x+2)}{3\times 1} + \frac{(x+2)^2}{3^2\times 2} + \frac{(x+2)^3}{3^3\times 3} + \ldots$.

A.a. Write down the general term. [1]

A.b. Find the interval of convergence. [13]

B. Solve the differential equation $(u + 3v^3)\frac{dv}{du} = 2v$, giving your answer in the form $u = f(v)$. [8]

The function f is defined by $f(x) = \ln(1 + \sin x)$.

When a scientist measures the concentration μ of a solution, the measurement obtained may be assumed to be a normally distributed random variable with mean μ and standard deviation 1.6.

A.a. Show that $f''(x) = \frac{-1}{1+\sin x}$. [4]

A.b. Determine the Maclaurin series for $f(x)$ as far as the term in x^4. [6]

A.c Deduce the Maclaurin series for $\ln(1 - \sin x)$ as far as the term in x^4 . [2]

A.d By combining your two series, show that $\ln \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \ldots$. [4]

A.e Hence, or otherwise, find $\lim_{x \to 0} \frac{\ln \sec x}{x\sqrt{x}}$. [2]

B.a He makes 5 independent measurements of the concentration of a particular solution and correctly calculates the following confidence interval for μ . [5]

$$[22.7 \, , 26.1]$$

Determine the confidence level of this interval.

B.b He is now given a different solution and is asked to determine a 95% confidence interval for its concentration. The confidence interval is required to have a width less than 2. Find the minimum number of independent measurements required. [5]

Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$.

a. Show that, for $n \geq 2$, $S_{2n} > S_n + \frac{1}{2}$. [3]

b. Deduce that $S_{2m+1} > S_2 + \frac{m}{2}$. [7]

c. Hence show that the sequence $\{S_n\}$ is divergent. [3]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$, where $y \neq 0$.

a. Find the general solution of the differential equation, expressing your answer in the form $f(x,\ y) = c$, where c is a constant. [3]

b. (i) Hence find the particular solution passing through the points $(1, \pm\sqrt{2})$. [5]

(ii) Sketch the graph of your solution and name the type of curve represented.

c. (i) Write down the particular solution passing through the points $(1,\ \pm 1)$. [3]

(ii) Give a geometrical interpretation of this solution in relation to part (b).

d. (i) Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x}$, where $xy \neq 0$. [12]

(ii) Find the particular solution passing through the point $(1,\ \sqrt{2})$.

(iii) Sketch the particular solution.

(iv) The graph of the solution only contains points with $|x| > a$.

Find the exact value of $a,\ a > 0$.

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a. Using a Taylor series, find a quadratic approximation for $f(x) = \sin x$ centred about $x = \frac{3\pi}{4}$. [4]

b. When using this approximation to find angles between 130° and 140°, find the maximum value of the Lagrange form of the error term. [7]

c. Hence find the largest number of decimal places to which $\sin x$ can be estimated for angles between 130° and 140°. [1]

d. Explain briefly why the same maximum value of error term occurs for $g(x) = \cos x$ centred around $\frac{\pi}{4}$ when finding approximations for angles between 40° and 50°. [3]

A machine fills containers with grass seed. Each container is supposed to weigh 28 kg. However the weights vary with a standard deviation of 0.54 kg. A random sample of 24 bags is taken to check that the mean weight is 28 kg.

A.a Assuming the series for e^x , find the first five terms of the Maclaurin series for [3]

$$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-x^2}{2}}.$$

A.b (i) Use your answer to (a) to find an approximate expression for the cumulative distributive function of $\mathrm{N}(0, 1)$. [6]

(ii) **Hence** find an approximate value for $\mathrm{P}(-0.5 \le Z \le 0.5)$, where $Z \sim \mathrm{N}(0, 1)$.

B.a State and justify an appropriate test procedure giving the null and alternate hypotheses. [5]

B.b What is the critical region for the sample mean if the probability of a Type I error is to be 3.5%? [7]

B.c If the mean weight of the bags is actually 28.1 kg, what would be the probability of a Type II error? [2]

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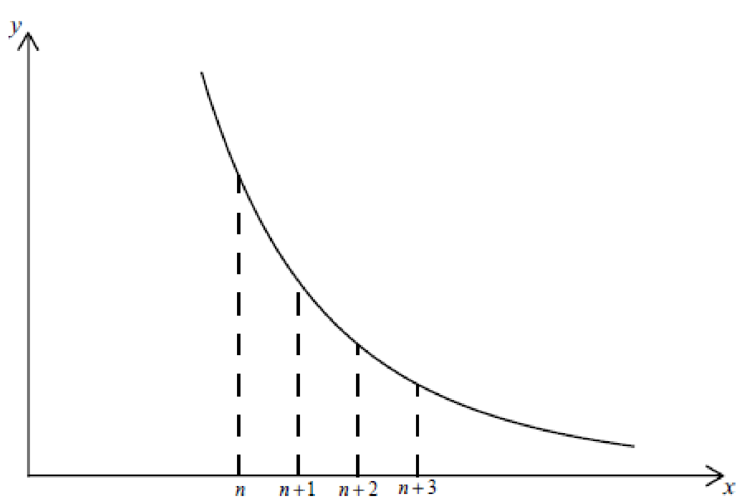
a. The diagram shows a sketch of the graph of $y = x^{-4}$ for $x > 0$. [5]



By considering this sketch, show that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=n+1}^{\infty} \frac{1}{r^4} < \int_n^{\infty} \frac{dx}{x^4} < \sum_{r=n}^{\infty} \frac{1}{r^4}.$$

b. Let $S = \sum_{r=1}^{\infty} \frac{1}{r^4}$. [8]

Use the result in (a) to show that, for $n \geq 2$, the value of S lies between

$$\sum_{r=1}^{n-1} \frac{1}{r^4} + \frac{1}{3n^3} \text{ and } \sum_{r=1}^{n} \frac{1}{r^4} + \frac{1}{3n^3} .$$

c. (i) Show that, by taking $n = 8$, the value of S can be deduced correct to three decimal places and state this value. [6]

(ii) The exact value of S is known to be $\frac{\pi^4}{N}$ where $N \in \mathbb{Z}^+$. Determine the value of N .

d. Now let $T = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^4}$. [3]

Find the value of T correct to three decimal places.

Consider the functions $f_n(x) = \sec^n(x),\ |x| < \frac{\pi}{2}$ and $g_n(x) = f_n(x) \tan x$.

a. Show that [5]

(i) $\frac{df_n(x)}{dx} = ng_n(x);$

(ii) $\frac{dg_n(x)}{dx} = (n+1)f_{n+2}(x) - nf_n(x).$

b. (i) Use these results to show that the Maclaurin series for the function $f_5(x)$ up to and including the term in x^4 is $1 + \frac{5}{2}x^2 + \frac{85}{24}x^4$. [14]

(ii) By considering the general form of its higher derivatives explain briefly why all coefficients in the Maclaurin series for the function $f_5(x)$ are either positive or zero.

(iii) Hence show that $\sec^5(0.1) > 1.02535$.

Consider the differential equation

$$\frac{dy}{dx} + y \sec x = x(\sec x - \tan x),\ \text{where } y = 3 \text{ when } x = 0.$$

a. Use Euler's method with a step length of 0.1 to find an approximate value for y when $x = 0.3$. [5]

b. (i) By differentiating the above differential equation, obtain an expression involving $\frac{d^2y}{dx^2}$. [8]

(ii) Hence determine the Maclaurin series for y up to the term in x^2 .

(iii) Use the result in part (ii) to obtain an approximate value for y when $x = 0.3$.

c. (i) Show that $\sec x + \tan x$ is an integrating factor for solving this differential equation. [11]

(ii) Solve the differential equation, giving your answer in the form $y = f(x)$.

(iii) Hence determine which of the two approximate values for y when $x = 0.3$, obtained in parts (a) and (b), is closer to the true value.

a. The function f is defined by $f(x)=\frac{e^x+e^{-x}}{2}$. [13]

(i) Obtain an expression for $f^{(n)}(x)$, the nth derivative of $f(x)$ with respect to x.

(ii) Hence derive the Maclaurin series for $f(x)$ up to and including the term in x^4 .

(iii) Use your result to find a rational approximation to $f\left(\frac{1}{2}\right)$.

(iv) Use the Lagrange error term to determine an upper bound to the error in this approximation.

b. Use the integral test to determine whether the series $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ is convergent or divergent. [9]

The random variable X has probability density function given by

$$f(x)=\begin{cases} xe^{-x}, & \text{for } x\geqslant 0,\\ 0, & \text{otherwise}\end{cases}.$$

A sample of size 50 is taken from the distribution of X.

a. Use l'Hôpital's rule to show that $\lim_{x\to\infty}\frac{x^3}{e^x}=0$. [3]

b. (i) Find $\mathrm{E}(X^2)$. [10]

(ii) Show that $\mathrm{Var}(X)=2$.

c. State the central limit theorem. [2]

d. Find the probability that the sample mean is less than 2.3. [2]

It is given that $(5x+y)\frac{dy}{dx}=(x+5y)$ and that when $x=0,\ y=2$.

a. Use Euler's method with step length 0.1 to find an approximate value of y when $x=0.4$. [5]

b.i. Show that $(5x+y)\frac{d^2y}{dx^2}=1-\left(\frac{dy}{dx}\right)^2$. [3]

b.ii. Show that $(5x+y)\frac{d^3y}{dx^3}=-5\frac{d^2y}{dx^2}-3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$. [4]

b.iii. Find the Maclaurin expansion for y up to and including the term in x^3. [5]

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a. Find the value of $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$. [6]

b. Find the interval of convergence of the infinite series [10]

$$\frac{(x+2)}{3\times 1}+\frac{(x+2)^2}{3^2\times 2}+\frac{(x+2)^3}{3^3\times 3}+\dots$$

c. (i) Find the Maclaurin series for $\ln(1+\sin x)$ up to and including the term in x^3. [12]

(ii) **Hence** find a series for $\ln(1-\sin x)$ up to and including the term in x^3.

(iii) Deduce, by considering the difference of the two series, that $\ln 3 \simeq \frac{\pi}{3}\left(1+\frac{\pi^2}{216}\right)$.

Consider the differential equation $\frac{dy}{dx}+y\tan x=2\cos^4 x$ given that $y=1$ when $x=0$.

(a) Solve the differential equation, giving your answer in the form $y=f(x)$.

(b) (i) By differentiating both sides of the differential equation, show that

$$\frac{d^2y}{dx^2}+y=-10\sin x\cos^3 x$$

(ii) Hence find the first four terms of the Maclaurin series for y.

(a) (i) Using l'Hôpital's rule, show that

$$\lim_{x\to\infty}\frac{x^n}{e^{\lambda x}}=0;\ n\in\mathbb{Z}^+,\ \lambda\in\mathbb{R}^+$$

(ii) Using mathematical induction on n, prove that

$$\int_0^\infty x^n e^{-\lambda x}dx=\frac{n!}{\lambda^{n+1}};\ n\in\mathbb{N},\ \lambda\in\mathbb{R}^+$$

(b) The random variable X has probability density function

$$f(x)=\begin{cases}\frac{\lambda^{n+1}x^n e^{-\lambda x}}{n!} x\geq 0, n\in\mathbb{Z}^+, \lambda\in\mathbb{R}^+\\ \text{otherwise}\end{cases}$$

Giving your answers in terms of n and λ, determine

(i) $E(X)$;

(ii) the mode of X.

(c) Customers arrive at a shop such that the number of arrivals in any interval of duration d hours follows a Poisson distribution with mean $8d$. The third customer on a particular day arrives T hours after the shop opens.

(i) Show that $P(T>t)=e^{-8t}\left(1+8t+32t^2\right)$.

(ii) Find an expression for the probability density function of T.

(iii) Deduce the mean and the mode of T.

Draw slope fields for the following cases for $-2\leqslant x\leqslant 2,\ -2\leqslant y\leqslant 2$

Explain what isoclines tell you about the slope field in the following case:

a.i. $\frac{\mathrm{d}y}{\mathrm{d}x} = 2.$ [2]

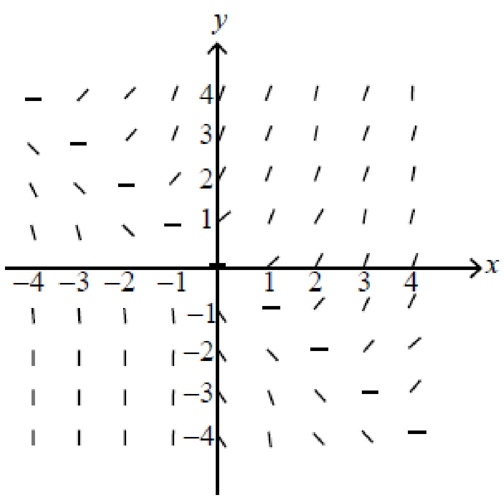
a.ii. $\frac{\mathrm{d}y}{\mathrm{d}x} = x + 1.$ [2]

a.iii. $\frac{\mathrm{d}y}{\mathrm{d}x} = x - 1.$ [2]

b.i. $\frac{\mathrm{d}y}{\mathrm{d}x} =$ constant. [1]

b.ii. $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$ [1]

c. The slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$ for $-4 \leqslant x \leqslant 4,\ -4 \leqslant y \leqslant 4$ is shown in the following diagram. [2]



Explain why the slope field indicates that the only linear solution is $y = -x - 1$.

d. Given that all the isoclines from a slope field of a differential equation are straight lines through the origin, find two examples of the differential equation. [4]